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A non-diagrammatic method for perturbative calculations in field theory

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Abstract. Feynman's diagrammatic approach to perturbative quantum field theory is not easily applied unless the interactions have simple power series expansions. In particular, when the interaction involves non-integer powers of the field, as happens when carrying out the so-called δ-expansion, the diagrammatic approach must be supplemented with some prescription for the analytic continuation of the exponents. Here we propose a new approach to perturbative field theory which bypasses Wick's theorem, and uses instead the fact that the joint probability distribution function for the fields at a finite set of points can be determined exactly from their expectation values, variances and mutual covariances. One can then calculate expectation values for products of operators at these points, or at least express them as finite-dimensional definite integrals. This technique is illustrated by calculating expectation values for non-polynomial $O(n)$-invariant operators.

Field theory calculations almost always reduce to some kind of perturbative expansion, the zeroth-order starting point being a Gaussian functional integral which can be done exactly. Higher-order corrections are then obtained by calculating expectation values of interaction terms and their products at different points in space. This is conventionally done using Feynman diagrams; however attempts to perform systematic calculations in the δ-expansion [1] led us to develop the non-diagrammatic techniques described in this paper. The reason for this is that in the δ-expansion one has to consider interaction terms which are not simple power series in the fields; they may involve logarithms, for example. To date the approach has been to use Feynman diagram techniques for interactions consisting of fields raised to an integer power, continuing from the integers to the reals and then differentiating with respect to the real exponent. The difficulty is that the continuation is not unique and we found that some naive prescriptions gave divergent (nonsensical) results. The new non-diagrammatic technique discussed later eliminates the need for analytic continuation and moreover can be applied to more general situations.

Our method is based on the observation that in a free Euclidean theory the field may be thought of as a set of correlated random variables with a multivariate Gaussian distribution. Consequently, the joint probability distribution of any finite subset of these variables can be determined exactly from their expectation values, variances and mutual covariances.

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To illustrate this point, let ϕ be a scalar field in a free Eucidean theory. The functional-integral prescription requires one to assign a weight $\mathcal{P}[\phi]$ to each configuration ϕ. Suppose we wish to calculate the expectation value of $\phi^{2\alpha}(\boldsymbol{x})$, where α is any positive real number. Writing

$$
\begin{aligned}
\langle\phi^{2\alpha}(\boldsymbol{x})\rangle &= \int \mathrm{d}[\phi]\, \phi^{2\alpha}(\boldsymbol{x})\mathcal{P}[\phi] \\
&= \int \mathrm{d}\phi_{\boldsymbol{x}} \int \mathrm{d}[\phi]\, \delta(\phi_{\boldsymbol{x}} - \phi(\boldsymbol{x}))\phi^{2\alpha}(\boldsymbol{x})\mathcal{P}[\phi] \\
&= \int \mathrm{d}\phi_{\boldsymbol{x}}\, \phi_{\boldsymbol{x}}^{2\alpha} P[\phi_{\boldsymbol{x}}]
\end{aligned}
\tag{1}
$$

where

$$
P(\phi_{\boldsymbol{x}}) \equiv \int \mathrm{d}[\phi]\, \mathcal{P}[\phi]\delta(\phi_{\boldsymbol{x}} - \phi(\boldsymbol{x})) \tag{2}
$$

we see it is possible to reduce the functional integral to a single integral with the marginal probability distribution (2) as weight function. Although (2) still seems to be a formidably difficult quantity to evaluate, in a free theory the action functional is quadratic and so the marginal distribution of ϕ at the point $\boldsymbol{x}$ is Gaussian. Its exact form can therefore be found if the mean and variance of the field is known.

The same idea goes over without any essential changes for the case of an n-component field $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$. The marginal distribution

$$
P(\boldsymbol{\phi}_{\boldsymbol{x}}) \equiv \int \mathrm{d}[\boldsymbol{\phi}]\, \mathcal{P}[\boldsymbol{\phi}]\delta^n(\boldsymbol{\phi}_{\boldsymbol{x}} - \boldsymbol{\phi}(\boldsymbol{x})) \tag{3}
$$

will be Gaussian with its exact form depending on the means and variances of each of the n components, and the $n(n-1)/2$ mutual covariances between pairs of distinct components.

Suppose now that the theory has an underlying $\mathrm{O}(n)$-invariance, which may however be broken by a linear coupling to an external source. Then the matrix of covariances between the components is $\mathrm{O}(n)$ invariant and has the form

$$
\langle\phi_a\phi_b\rangle - \bar{\phi}_a\bar{\phi}_b = \sigma^2\delta_{ab} \tag{4}
$$

where $\bar{\phi}_a \equiv \langle\phi_a\rangle$ denotes the expectation value of each component, and σ^2 is their common variance; the covariance between any distinct components must vanish. (In fact the covariance matrix is just $G_{ab}(\boldsymbol{x}, \boldsymbol{x})$ where G is the Green function of the theory.) The probability distribution for $\boldsymbol{\phi}$ at the point $\boldsymbol{x}$ must then be

$$
P(\boldsymbol{\phi}_{\boldsymbol{x}}) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\|\boldsymbol{\phi}_{\boldsymbol{x}} - \bar{\boldsymbol{\phi}}\|^2/2\sigma^2\right\}. \tag{5}
$$

The expectation value of any function of $\boldsymbol{\phi}$ at the point $\boldsymbol{x}$ can be obtained by integrating over $\boldsymbol{\phi}_{\boldsymbol{x}}$ with this weight.

In this example, the $\mathrm{O}(n)$ symmetry ensures that operators of interest will generally have the form $F(\|\boldsymbol{\phi}\|)$, where $\|\boldsymbol{\phi}\|$ denotes the magnitude of the vector $\boldsymbol{\phi}$,

$$
\|\boldsymbol{\phi}\|^2 = \boldsymbol{\phi}\cdot\boldsymbol{\phi} \equiv \sum_{a=1}^{n}(\phi_a)^2. \tag{6}
$$

One is often interested in the expectation values of these operators, or the products of such operators evaluated at different points in space, given that the expectation value of the field itself is specified.

If α is any non-negative real number, then the expectation value of $\|\bar{\phi}(x)\|^{2\alpha}$ is

$$\Psi_n(\alpha,\bar{\phi},\sigma^2) \equiv \int \mathrm{d}^n\phi\,(\bar{\phi}\cdot\bar{\phi})^\alpha P(\phi). \tag{7}$$

For $n = 1$, this integral can be evaluated directly using standard results involving parabolic cylinder functions [2]. One finds that

$$\Psi_1(\alpha,\bar{\phi},\sigma^2) = \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})}(2\sigma^2)^\alpha M(-\alpha,\tfrac{1}{2};-\|\bar{\phi}\|^2/2\sigma^2) \tag{8}$$

where $M(a,b;z)$ is a confluent hypergeometric function with power series expansion

$$M(a,b;z) \equiv 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{z^3}{3!} + \cdots. \tag{9}$$

When α is an integer, the expansion terminates and (8) agrees with the results obtained using Feynman diagrams.

Results for n-component fields ($n > 1$) can be calculated in the following manner. Because of the $\mathrm{O}(n)$ symmetry, one can assume without loss of generality that the only component of ϕ with a non-vanishing expectation value is ϕ_1. Defining a new variable $y = \sum_{a=2}^n(\phi_a)^2$, the remaining $\mathrm{O}(n-1)$ symmetry of the integrand allows us to rewrite the measure in (7) as

$$\mathrm{d}^n\phi = \frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)}y^{(n-3)/2}\,\mathrm{d}y\,\mathrm{d}\phi_1. \tag{10}$$

Setting $y \equiv (\phi_1)^2 t$ allows one to perform the ϕ_1 integral directly, using the earlier result for Ψ_1. Performing the remaining integral one then finds that

$$\Psi_n(\alpha,\bar{\phi},\sigma^2) = \frac{\Gamma(\alpha+\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)}(2\sigma^2)^\alpha M(-\alpha,\tfrac{1}{2}n;-\|\bar{\phi}\|^2/2\sigma^2). \tag{11}$$

A more challenging exercise is to work out expectation values of products of operators at different points $x^{(i)}$ in space. If we know the marginal expectation value and variance of the field at each specified point, as well as the covariance between the field variables at any pair of these points (these can be obtained from the Green function), then we can derive the exact form of the joint probability distribution function for the variables $\phi(x^{(i)})$ since we know that these are Gaussian. Thus, the expectation values of the products of any functions of these variables can be represented as an ordinary definite integral. The only question is whether this integral can be evaluated exactly.

For example, suppose we have just two points $x^{(1)}, x^{(2)}$, and that the fields at these points are denoted $\phi^{(1)}$ and $\phi^{(2)}$ respectively. We assume that these are known to have expectation values $\bar{\phi}^{(1)}$ and $\bar{\phi}^{(2)}$ respectively, and that all the components have variance σ^2. The joint probability distribition of $\phi^{(1)}$ and $\phi^{(2)}$ is then completely determined by the matrix of covariances between their components, which is exactly the same as the Green function $G_{ab}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})$.

Since the $\mathrm{O}(n)$ symmetry is broken only by a linear source term, the covariance matrix will have the $\mathrm{O}(n)$-invariant form

$$\left\{\langle\phi_a^{(1)}\phi_b^{(2)}\rangle - \bar{\phi}_a^{(1)}\bar{\phi}_b^{(2)}\right\} = \rho\sigma^2\delta_{ab} \tag{12}$$

where ρ is a real number between -1 and 1. The joint probability distribution function for $\phi^{(1)}$ and $\phi^{(2)}$ will therefore be

$$P(\phi^{(1)}, \phi^{(2)}) = \left(\frac{(1-\rho^2)^{-1/2}}{2\pi\sigma^2}\right)^n \exp\left\{-\frac{(1-\rho^2)^{-1}}{2\sigma^2}\right. \\ \left.\times\left[\|\Delta\phi^{(1)}\|^2 + \|\Delta\phi^{(2)}\|^2 - 2\rho\Delta\phi^{(1)}\cdot\Delta\phi^{(2)}\right]\right\} \tag{13}$$

where $\Delta\phi^{(i)} \equiv \phi^{(i)} - \bar{\phi}^{(i)}$.

The expectation value of any function of $\phi^{(1)}$ and $\phi^{(2)}$ can now be expressed as the $2n$-dimensional definite integral of the function weighted with the probability density (13). For example, suppose that α and β are non-negative real numbers, and that we are interested in the expectation value of the product $\|\phi^{(1)}\|^{2\alpha}\|\phi^{(2)}\|^{2\beta}$. This is just

$$\Xi_n(\rho, \sigma^2; \alpha, \beta; \bar{\phi}^{(1)}, \bar{\phi}^{(2)}) \equiv \int \mathrm{d}^n\phi^{(1)}\,\mathrm{d}^n\phi^{(2)}\,\|\phi^{(1)}\|^{2\alpha}\|\phi^{(2)}\|^{2\beta}P(\phi^{(1)}, \phi^{(2)}). \tag{14}$$

Although there seems to be no way of doing this integral directly, it can be evaluated as a power series in ρ. (Such an expansion is reasonable, since $|\rho|$ is always less than unity and decreases rapidly as the distance between $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ increases.)

It is easy to show that

$$\frac{\partial\Xi_n}{\partial\rho} = \sigma^2\sum_{a=1}^{n}\frac{\partial^2\Xi_n}{\partial\bar{\phi}_a^{(1)}\partial\bar{\phi}_a^{(2)}}. \tag{15}$$

Moreover, when the correlation coefficient ρ vanishes, both $P(\phi^{(1)}, \phi^{(2)})$ and the integral (14) factorize into two independent terms. Consequently, for $\rho = 0$ we have

$$\Xi_n(0, \sigma^2; \alpha, \beta; \bar{\phi}^{(1)}, \bar{\phi}^{(2)}) = \int \mathrm{d}^n\phi^{(1)}\,\|\phi^{(1)}\|^{2\alpha}P(\phi^{(1)})\int \mathrm{d}^n\phi^{(2)}\,\|\phi^{(2)}\|^{2\beta}P(\phi^{(2)}) \\ = \Psi_n(\alpha, \bar{\phi}^{(1)}, \sigma^2)\Psi_n(\beta, \bar{\phi}^{(2)}, \sigma^2). \tag{16}$$

One can now solve the differential equation (15) subject to the boundary condition (16), to obtain

$$\Xi_n(\rho, \sigma^2; \alpha, \beta; \bar{\phi}^{(1)}, \bar{\phi}^{(2)}) = (2\sigma^2)^{(\alpha+\beta)}\frac{\Gamma(\alpha+\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)}\frac{\Gamma(\beta+\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)} \\ \times\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}C(k,l,m,r)\rho^{m+2r} \\ \times\left[-\frac{\|\bar{\phi}^{(1)}\|^2}{2\sigma^2}\right]^k\left[-\frac{\|\bar{\phi}^{(2)}\|^2}{2\sigma^2}\right]^l\left[-\frac{\bar{\phi}^{(1)}.\bar{\phi}^{(2)}}{\sigma^2}\right]^m \tag{17}$$

where the coefficients C have the form

$$C(k,l,m,r) = \frac{(-\alpha)_{k+m+r}\,(-\beta)_{l+m+r}\,(k+l+m+\frac{1}{2}n)_r}{k!\,l!\,m!\,r!\,(\frac{1}{2}n)_{k+m+r}(\frac{1}{2}n)_{l+m+r}}. \tag{18}$$

Here we are using Pochhammer's notation

$$(z)_n \equiv z(z+1)(z+2)\ldots(z+n-1). \tag{19}$$

In fact the identity (17) is the special case of a quite general result. If u is a random vector whose components $(u_1,\ldots,u_m)$ have a multivariate Gaussian distribution with means $(\bar{u}_1,\ldots,\bar{u}_m)$ and covariance matrix

$$A_{ij} \equiv \langle u_i u_j\rangle - \bar{u}_i\bar{u}_j \tag{20}$$

then its probability distribution function is

$$P(\boldsymbol{u}) = (2\pi)^{-m/2}(\det A)^{-1/2}\exp\left\{-\tfrac{1}{2}\sum_{i,j}(u_i-\bar{u}_i)[A^{-1}]_{ij}(u_j-\bar{u}_j)\right\}. \tag{21}$$

One then finds that

$$\frac{\partial P}{\partial A_{ij}} = \frac{1}{2}\frac{\partial^2 P}{\partial\bar{u}_i\partial\bar{u}_j}. \tag{22}$$

Since the probability distribution is Gaussian, the expectation value of any function $H(\boldsymbol{u})$ can be expressed as a function of the mean vector $\bar{\boldsymbol{u}}$ and the covariance matrix A;

$$\begin{aligned}\langle H(\boldsymbol{u})\rangle &= \int \mathrm{d}^m\boldsymbol{u}\, P(\boldsymbol{u})H(\boldsymbol{u}) \\ &\equiv \bar{H}(\boldsymbol{u},A).\end{aligned} \tag{23}$$

It is clear from this definition that $\bar{H}$ obeys the same differential equation (22) as $P(\boldsymbol{u})$. Thus, if we know the form of $\bar{H}$ for a particular choice of A_{ij}, we can (in principle) solve this differential equation to determine $\langle H\rangle$ for arbitrary u and A. Knowledge of $\langle H\rangle$ when the components u_i are uncorrelated can therefore be used to determine its value for arbitrary correlations. The derivation of (17) was a special case of this argument.

We now show how these results can be used to obtain an expansion of the effective action in powers of some (natural or artificial) perturbation parameter α. Suppose that a Euclidean theory has classical action $S_\alpha[\phi]$, which is quadratic in ϕ when $\alpha = 0$. Introducing a source $J(\boldsymbol{x})$, we define the generating functionals

$$Z_\alpha[J] \equiv \int \mathrm{d}[\phi]\exp\left\{-S_\alpha[\phi] + \int \mathrm{d}^d\boldsymbol{x}\,\boldsymbol{J}\cdot\boldsymbol{\phi}\right\} \tag{24}$$

and

$$W_\alpha[J] \equiv \ln\{Z_\alpha[J]/Z_\alpha[0]\}. \tag{25}$$

We suppose that for a given configuration $\bar{\phi}$ there is a corresponding configuration $J_{\alpha,\bar{\phi}}$ of the source such that

$$\bar{\phi} = \left(\frac{\delta W_\alpha}{\delta J(x)}\right)_\alpha \Bigg|_{J=J_{\alpha,\bar{\phi}}}. \tag{26}$$

In fact, $\bar{\phi}$ is just the expectation value of the field when the action includes a coupling to the source $J_{\alpha,\bar{\phi}}$.

The effective action of the configuration $\bar{\phi}$ can now be defined as

$$\Gamma_\alpha[\bar{\phi}] = \int \mathrm{d}^d x \, \bar{\phi} \cdot J_{\alpha,\bar{\phi}} - W_\alpha[J_{\alpha,\bar{\phi}}]. \tag{27}$$

This quantity can be expanded as a Taylor series in α:

$$\Gamma_\alpha[\bar{\phi}] = \Gamma^{(0)}[\bar{\phi}] + \alpha\Gamma^{(1)}[\bar{\phi}] + \frac{\alpha^2}{2}\Gamma^{(2)}[\bar{\phi}] + \cdots \tag{28}$$

where $\Gamma^{(k)} \equiv \partial^k \Gamma_\alpha / \partial\alpha^k|_{\alpha=0}$. The leading term $\Gamma_0[\phi]$ coincides with the free action, while the higher derivatives can be calculated using our earlier results.

For example

$$\left(\frac{\partial \Gamma_\alpha}{\partial \alpha}\right)_{\bar{\phi}} = \int \mathrm{d}^d x \, \bar{\phi} \cdot \left(\frac{\partial J_{\alpha,\bar{\phi}}}{\partial \alpha}\right)_{\bar{\phi}} - \left(\frac{\partial W_\alpha}{\partial \alpha}\right)_{J=J_{\alpha,\bar{\phi}}}$$
$$- \int \mathrm{d}^d x \left(\frac{\partial J_{\alpha,\bar{\phi}}(x)}{\partial \alpha}\right)_{\bar{\phi}} \cdot \left(\frac{\delta W_\alpha}{\delta J(x)}\right)_\alpha. \tag{29}$$

The first and last terms cancel thanks to equation (26). Evaluated at $\alpha = 0$, this expression yields

$$\Gamma^{(1)}[\bar{\phi}] = \langle S^{(1)} \rangle_{\bar{\phi}} - \langle S^{(1)} \rangle_0 \tag{30}$$

where the subscripts indicate that appropriate source terms have been included so that the expectation value of the field is $\bar{\phi}$ or 0 respectively, and $S^{(k)} \equiv \partial^k S_\alpha / \partial \alpha^k|_{\alpha=0}$.

In most cases of interest, the only part of the action S_α which will be sensitive to the value of α is the potential term $\int \mathrm{d}^d x \, V_\alpha(\phi)$. We can then write

$$\Gamma^{(1)}[\bar{\phi}] = \frac{\partial}{\partial \alpha}\left(\int \mathrm{d}^d x \, \langle V_\alpha \rangle_{\bar{\phi}}\right)\Bigg|_{\alpha=0}. \tag{31}$$

If $V_\alpha(\phi)$ is $\mathrm{O}(n)$-invariant then it can normally by written as a linear combination of powers of $||\phi||$; in such cases our earlier result (8) yields an explicit expression for the expectation value in (31). Note that this method can be applied to potentials such as $||\phi||^{2(1+\alpha)}$ in which the expansion parameter is an exponent, as well as more conventional potentials in which α acts as a coupling constant. Our approach can therefore be used to perform δ-expansions [1].

One can also calculate the higher derivatives of Γ_α. To obtain the next term, we observe that

$$\left(\frac{\partial^2\Gamma_\alpha}{\partial\alpha^2}\right)_{\bar\phi} = -\left(\frac{\partial}{\partial\alpha}\right)_{\bar\phi}\left(\frac{\partial W_\alpha}{\partial\alpha}\right)_{J=J_{\alpha,\bar\phi}}$$
$$= \left(\frac{\partial^2 W_\alpha}{\partial\alpha^2}\right)_{J=J_{\alpha,\bar\phi}} - \int \mathrm{d}^d\boldsymbol{x}\left(\frac{\partial J_{\alpha,\bar\phi}(\boldsymbol{x})}{\partial\alpha}\right)_{\bar\phi}\left(\frac{\delta\partial W_\alpha}{\delta J(\boldsymbol{x})\partial\alpha}\right)_{\bar\phi}. \tag{32}$$

Furthermore,

$$\left(\frac{\delta\partial W_\alpha}{\delta J(\boldsymbol{x})\partial\alpha}\right)_{\bar\phi} = \int \mathrm{d}^d\boldsymbol{x}'\left(\frac{\delta\bar\phi(\boldsymbol{x}')}{\delta J(\boldsymbol{x})}\right)_\alpha\left(\frac{\delta}{\delta\bar\phi(\boldsymbol{x}')}\right)_\alpha\left(\frac{\partial W_\alpha}{\partial\alpha}\right)_{J=J_{\alpha,\bar\phi}}$$
$$= -\int \mathrm{d}^d\boldsymbol{x}'\left(\frac{\delta^2 W_\alpha}{\delta J(\boldsymbol{x})\delta J(\boldsymbol{x}')}\right)_\alpha\left(\frac{\delta}{\delta\bar\phi(\boldsymbol{x}')}\right)_\alpha\left(\frac{\partial\Gamma_\alpha}{\partial\alpha}\right)_{\bar\phi} \tag{33}$$

while the familiar identity $J_{\alpha,\bar\phi} = \delta\Gamma_\alpha/\delta\bar\phi$ implies that

$$\left(\frac{\partial J_{\alpha,\bar\phi}(\boldsymbol{x})}{\partial\alpha}\right)_{\bar\phi} = \left(\frac{\partial}{\partial\alpha}\right)_{\bar\phi}\left(\frac{\delta\Gamma_\alpha}{\delta\bar\phi}\right)_\alpha = \left(\frac{\delta}{\delta\bar\phi}\right)_\alpha\left(\frac{\partial\Gamma_\alpha}{\partial\alpha}\right)_{\bar\phi}. \tag{34}$$

One can also show that

$$\left(\frac{\partial^2 W_\alpha}{\partial\alpha^2}\right)_{J=J_{\alpha,\bar\phi}} = \left\langle\frac{\partial^2 S_\alpha}{\partial\alpha^2} - \left(\frac{\partial S_\alpha}{\partial\alpha}\right)^2\right\rangle_{\bar\phi} + \left[\left\langle\frac{\partial S_\alpha}{\partial\alpha}\right\rangle_{\bar\phi}\right]^2$$
$$-\left\langle\frac{\partial^2 S_\alpha}{\partial\alpha^2} - \left(\frac{\partial S_\alpha}{\partial\alpha}\right)^2\right\rangle_0 - \left[\left\langle\frac{\partial S_\alpha}{\partial\alpha}\right\rangle_0\right]^2. \tag{35}$$

Setting $\alpha = 0$, and using the fact that $\delta^2 W/\delta J^2|_{\alpha=0}$ is just the Green function, we obtain the result

$$\Gamma^{(2)}[\bar\phi] = \left\langle S^{(2)} - (S^{(1)})^2\right\rangle_{\bar\phi} + \left[\langle S^{(1)}\rangle_{\bar\phi}\right]^2 - \left\langle S^{(2)} - (S^{(1)})^2\right\rangle_0 - \left[\langle S^{(1)}\rangle_0\right]^2$$
$$+\int \mathrm{d}^d\boldsymbol{x}\,\mathrm{d}^d\boldsymbol{x}'\,\frac{\delta\Gamma^{(1)}}{\delta\bar\phi(\boldsymbol{x})}\cdot G(\boldsymbol{x},\boldsymbol{x}')\cdot\frac{\delta\Gamma^{(1)}}{\delta\bar\phi(\boldsymbol{x}')}. \tag{36}$$

If the only α-dependent part of S_α is the potential term, then

$$\left\langle S^{(2)} - (S^{(1)})^2\right\rangle + \langle S^{(1)}\rangle^2$$
$$= \int \mathrm{d}^d\boldsymbol{x}\left\langle V^{(2)}_{\boldsymbol{x}}\right\rangle + \int \mathrm{d}^d\boldsymbol{x}\,\mathrm{d}^d\boldsymbol{x}'\left[\left\langle V^{(1)}_{\boldsymbol{x}}\right\rangle\left\langle V^{(1)}_{\boldsymbol{x}'}\right\rangle - \left\langle V^{(1)}_{\boldsymbol{x}}V^{(1)}_{\boldsymbol{x}'}\right\rangle\right] \tag{37}$$

where $V^{(k)} \equiv \partial^k V_\alpha/\partial\alpha^k|_{\alpha=0}$. The expectation values in the integrands can be calculated using our earlier results, at least for $O(n)$-invariant theories, provided that we know the matrix of correlation coefficients between $\phi_a(\boldsymbol{x})$ and $\phi_b(\boldsymbol{x}')$. If the free action has the form

$$S^{(0)}[\phi] = \tfrac{1}{2} \int \mathrm{d}^d \boldsymbol{x} \, [(\nabla\phi)^2 + \mu^2 \phi \cdot \phi] \tag{38}$$

then the correlation coefficient is found to be just

$$\rho(\boldsymbol{x}, \boldsymbol{x}') = \Delta(r)/\Delta(0) \tag{39}$$

where $r = \|\boldsymbol{x} - \boldsymbol{x}'\|$,

$$\begin{aligned} \Delta(r) &\equiv \frac{1}{n} \left\{ \langle \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{x}') \rangle - \bar{\phi}(\boldsymbol{x}) \cdot \bar{\phi}(\boldsymbol{x}') \right\} \\ &= (2\pi)^{-d/2} \left(\frac{\mu}{r} \right)^{d/2-1} K_{d/2-1}(\mu r) \end{aligned} \tag{40}$$

and where $K_{d/2-1}$ is an associated Bessel function [2]. The divergence of $\Delta(0)$ does not present any problem if a suitable regularization scheme is used. (For example, one might replace $\Delta(0)$ by $\Delta(\epsilon)$ and eventually take the limit $\epsilon \to 0$.)

In conclusion, we remark that similar procedures can be used to obtain expressions for the higher-order terms in the effective action. In general, the kth derivative of Γ_α will reduce to a (kd)-dimensional definite integral.

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